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Quantum R-matrices and factorization problems

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The authors are happy to dedicate this paper to Professor I.M. Gel'fand on his 75th birthday

Abstract. *A relation between quantum R-matrices and certain factorization problem in Hopf algebras is established. A definition of dressing transformation in the quantum case is also given.*

The term "Quantum groups" has been recently proposed by V. Drinfeld [3] to cover a specific class of Hopf algebras that are intrinsically connected with the Quantum Inverse Scattering Method [1, 2]. As a matter of fact, the invention of Q.I.S.M. has provided a vast source of new examples for the theory of Hopf algebras. In this respect we should first of all mention of definition of quantized enveloping algebras of simple Lie algebras given by Drinfeld [4] and Jimbo [5] following the work of Kulish and Reshetikhin [6] that deals with the $sl(2)$ case. An alternative approach which keeps much closer to the original ideas of Q.I.S.M. has been developed by Faddeev, Reshetikhin and Takhtajan in [7].

The theory of Quantum groups has as its semi-classical counterpart the theory of Poisson Lie group, that is Lie groups equipped with Poisson bracket such that group multiplication is a Poisson mapping. The theory of Poisson Lie groups is also relatively new [8] and was motivated both by its relation to Quantum groups and by applications to an important class of classical integrable systems on 1-dimensional lattices described by difference Lax equations [9].

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The commutation relations in quantum groups are expressed by means of the so-called quantum R-matrices which are of fundamental importance for the theory. Their semi-classical counterparts are called classical r-matrices and serve to determine the Poisson bracket relations on Poisson Lie group referred to above. There exists also a profound connection between classical r-matrices and certain factorization problems in Lie group. In typical applications these latter are reduces to matrix Riemann-Hilbert problems that are so crucial to the study of solutions of Lax equations [9]. Thus classical r-matrices fulfil the double task of providing both natural Poisson brackets for non-linear Lax equations and also analytical tools that may be used to obtain their solutions. They also enter the definition of the so-called dressing transformations that play an important role in the theory [9].

The aim of the present paper is to establish a similar relation between quantum R-matrices and certain factorization problems in Hopf algebras. We also give a definition of dressing transformations in the quantum case. Connections with quantum integrable systems are presently under study and will be described in a separate publication.

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1. FACTORIZABLE LIE BIALGEBRAS

The present section exposes known facts on classical r-matrices in a way that is suited for generalizations to the quantum case.

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ its dual. A Lie algebra structure $[\ .\ ]_*$ on $\mathfrak{g}^*$ defines a map

$$\varphi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g} : \langle \varphi(X), f \wedge g \rangle = \langle X, [f, g]_* \rangle. \tag{1.1}$$

Lie brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$ are said to be compatible if φ is a 1-cocycle on $\mathfrak{g}$. i.e.

$$X \varphi(Y) - Y \varphi(X) - \varphi([X, Y]) = 0,$$

the action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$ being given by

$$X \cdot Y \wedge Z = ad\, \Delta(X) \cdot Y \wedge Z = [1 \otimes X + X \otimes 1, Y \otimes Z - Z \otimes Y]. \tag{1.2}$$

A pair $(\mathfrak{g}, \mathfrak{g}^*)$ with compatible Lie brackets is called a Lie bialgebra [8].

We shall be dealing with an apparently very special class of Lie bialgebras. (However, as we shall see, any Lie bialgebra is embedded canonically into a Lie bialgebra that falls into this class).

Fix an element $r \in \mathfrak{g} \otimes \mathfrak{g}$. We associate with it a trivial 2-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$

(1.3) $$\varphi_r(X) - [r, \Delta(X)] = [r, X \otimes 1 + 1 \otimes X]$$

which gives rise to a would-be commutator map $[\ ,\ ]_*: \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ defined by (1). Let us associate with r a linear operator

(1.4) $$r : \mathfrak{g}^* \to \mathfrak{g} : f \mapsto \langle f \otimes id, r \rangle.$$

Its adjoint is given by

(1.5) $$r^* : f \mapsto \langle id \otimes f, r \rangle = \langle f \otimes id, P(r) \rangle$$

where P is the permutation operator in $\mathfrak{g} \otimes \mathfrak{g}$, $P(X \otimes Y) = Y \otimes X$. The bracket $[\ ,\ ]_*$ is given by

(1.6) $$[f, g]_* = ad^* r(f) \cdot g + ad^* r^*(g) \cdot f$$

In order to define a Lie algebra structure on $\mathfrak{g}^*$ the bracket (6) must be skew and satisfy the Jacobi identity. Put

(1.7) $$I = r + P(r)$$

The skew symmetry of (6) is equivalent to the condition that

(1.8) $$[I, \Delta X] = [r + P(r), \Delta X] = 0$$

for all $X \in \mathfrak{g}$.

Put

(1.9) $$B_r = [r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{13}, r_{23}].$$

Here as usual r_{12} is the image of $r \in \mathfrak{g} \otimes \mathfrak{g}$ under the mapping $\mathfrak{g} \otimes \mathfrak{g} \to \to \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ which sends $a \otimes b$ into $a \otimes b \otimes 1$ and similarly for r_{13}, r_{23}.

PROPOSITION 1.1. If r satisfies (1.8) and moreover $B_r = 0$, expression (1.6) is a Lie bracket on $\mathfrak{g}^*$ that makes $(\mathfrak{g}, \mathfrak{g}^*)$ a Lie bialgebra.

Equation

(1.10) $$[r_{12}, r_{23}] + [r_{13}, r_{23}] + [r_{12}, r_{13}] = 0$$

is called the classical Yang-Baxter identity.

An obvious way to satisfy (8) is to choose a skew r, i.e. such that

$$(1.11) \qquad r + P(r) = 0.$$

Condition (11) is usually called the classical unitarity condition. However, we shall be interested in the opposite case.

DEFINITION 1.1. *The Lie bialgebra described in Proposition 1.1 is called factorizable if the linear map $I : \mathfrak{g}^* \to \mathfrak{g}$ defined by the kernel (7) is a linear isomorphism.* ■

Due to condition (8) this map is clarly $\mathfrak{g}$-equivariant. Hence the inverse map I^{-1} defines a nondegenerate invariant inner product on $\mathfrak{g}$

$$(1.12) \qquad \langle X, Y\rangle = \langle I^{-1}(X), Y\rangle.$$

We may identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the pairing (12) and speak of two different Lie algebra structures on the same linear space. (This point of view has advantages in the study of integrable systems and is adopted in [9, 10]).

To understand the meaning of the Yang-Baxter identity (10) let us rewrite it in operator form. A short calculation yields that (10) is equivalent to

$$(1.13) \qquad [rX, rY] - r([X, rY] - [r^*X, Y]) = 0$$

which means simply that $r : \mathfrak{g}^* \to \mathfrak{g}$ is a Lie algebra homomorphism. Observe now that $r_- = -P(r)$ defines the same Lie bracket on $\mathfrak{g}^*$, satisfies (10) and hence gives another Lie algebra homomorphism $r_- : \mathfrak{g}^* \to \mathfrak{g}$. We shall sometimes write r_+ instead of r. Let us consider the mappings

$$(1.14) \qquad \mathfrak{g}^* \xrightarrow{r_+ \oplus r_-} \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{(X, Y) \mapsto X - Y} \mathfrak{g}.$$

Since $r_+ - r_- = I$ the composition map coincides with I. Hence we get

PROPOSITION 1.2. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra. Then any element $X \in \mathfrak{g}$ admits a unique decomposition

$$(1.15) \qquad X = X_+ - X_-$$

with $(X_+, X_-) \in Im\ (r_+ \oplus r_-) \subset \mathfrak{g} \oplus \mathfrak{g}$. We refer the reader to [10] for a precise description of this image in terms of the Cayley transform of r.

Note. It is sometimes more convenient to deal with skew r-matrices. If we put

$$r_0 = r_+ - 1/2\, I = r_- + 1/2\, I$$

then

$$r_0 = -r_0^*.$$

However, r_0 now satisfies

$$[r_0 f, r_0 g] - r_0([r_0 f, g] + [f, r_0 g]) = -\frac{1}{4}[If, Ig]. \tag{1.16}$$

Equation (16) is called the modified Yang-Baxter equation [9, 10].

Let $U(\mathfrak{g})$, $U(\mathfrak{g}^*)$ by the universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{g}^*$, respectively. We endow them with the usual Hopf algebra structure. The comultiplication Δ is given by

$$\Delta X = X \otimes 1 + 1 \otimes X \tag{1.17}$$

and the antipode is

$$S(X) = -X \tag{1.18}$$

on the generators of degree 1.

Let us consider the following chain of mappings

$$U(\mathfrak{g}^*) \xrightarrow{\Delta} U(\mathfrak{g}^*) \otimes U(\mathfrak{g}^*) \xrightarrow{r_+ \otimes r_-} U(\mathfrak{g}) \otimes U(\mathfrak{g}) \xrightarrow{m\,(id \otimes S)} U(\mathfrak{g}) \tag{1.19}$$

where $m : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ is the multiplication map.

PROPOSITION 1.3. *The composition map $U(\mathfrak{g}^*) \to U(\mathfrak{g})$ is a linear isomorphism induced by $I : \mathfrak{g}^* \to \mathfrak{g}$. Any element $x \in U(\mathfrak{g})$ admits a unique representation*

$$x = \sum_i x_+^i S(x_-^i) \tag{1.20}$$

where

$$\sum x_+^i \otimes x_-^i = (r_+ \otimes r_-)\,\Delta\,(I^{-1}(x)) \tag{1.21}$$

lies in the range of $(r_+ \otimes r_-)\,\Delta$. ■

The next formula relates multiplications in $U(\mathfrak{g}^*)$ and $U(\mathfrak{g})$ and may be regarded as a generalization of (6). Put

$$x * y = I(I^{-1}(x) \cdot I^{-1}(y)). \tag{1.22}$$

The product * is the product in $U(\mathfrak{g}^*)$ pushed forward to $U(\mathfrak{g})$ by means of I.

PROPOSITION 1.4.

(1.23) $$x * y = \sum_i x_+^i \, y \, S(x_-^i).$$

Let us explain now the relation of the representations (20), (15) to the factorization problems in Lie groups. Let G, G^* be the local Lie groups corresponding to $\mathfrak{g}$, $\mathfrak{g}^*$, respectively. We may regard them as consisting of products of exponentials $g = e^x$ which lie in an appropriate completion of $U(\mathfrak{g})$ ($U(\mathfrak{g}^*)$) and satisfy $\Delta g = g \otimes g$, $Sg = g^{-1}$. Hence (20) implies that there is a unique decomposition for an element $g \in G$

(1.24) $$g = g_+ \, g_-^{-1}$$

with $(g_+, g_-) \in Im(r_+ \times r_-) \subset G \times G$. Formula (23) yields the following relation between the multiplications in G, G^*. Put $g \circ h = I\,(I^{-1}(g) \cdot I^{-1}(h))$ where I is the local homeomorphism between G^* and G induced by (7). Then

(1.25) $$g \circ h = g_+ \, h g_-^{-1}.$$

So far our discussion remained formal, since we did not give any examples of factorizable Lie bialgebras. An ample source of such examples is provided by the following construction [8].

THEOREM. *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be an arbitrary Lie bialgebra.*

(i) There exists a unique Lie algebra structure on $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ such that $\mathfrak{g}$, $\mathfrak{g}^* \subset \mathfrak{d}$ are Lie subalgebras and the natural pairing on $\mathfrak{d}$

(1.26) $$\langle (X, f), (Y, g) \rangle = g(X) + f(Y)$$

is an $\mathfrak{d}$-invariant.

(ii) Let P, P^* be the canonical projection operators onto $\mathfrak{g}$, $\mathfrak{g}^*$ in the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$. We may regard P, P^* as elements of $\mathfrak{d} \otimes \mathfrak{d}$ (in fact, P is the image of the canonical element in $\mathfrak{g} \otimes \mathfrak{g}^*$ under the natual embedding $\mathfrak{g} \otimes \mathfrak{g}^* \subset \mathfrak{d} \otimes \mathfrak{d}$). Put $(r\mathfrak{d})_+ = P$, $(r\mathfrak{d})_- = -P^*$. The r-matrices $(r\mathfrak{d})_\pm$ satisfy (10); moreover,

$$(r\mathfrak{d})_+ - (r\mathfrak{d})_- = I$$

is the identity operator. Hence they equip $(\mathfrak{d}, \mathfrak{d}^*)$ with the structure of a factorizable Lie bialgebra. Note that

$$\mathfrak{d}^* \simeq \mathfrak{g} \oplus \mathfrak{g}^*$$

as a Lie algebra. ∎

We shall refer to $(\mathfrak{d}, \mathfrak{d}^*)$ as the double of $(\mathfrak{g}, \mathfrak{g}^*)$. The theorem above was first stated in [8], however, with no special emphasis on the factorization properties. The notion of the double of a Lie bialgebra was then systematically used in [9].

It is interesting to notice that if $(\mathfrak{g}, \mathfrak{g}^*)$ is already a factorizable Lie bialgebra its double may be described more explicitly.

PROPOSITION 1.5. *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra. Then $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ as a Lie algebra.* ■

Indeed, observe that there are natural embeddings

$$\mathfrak{g} \to \mathfrak{d} : X \mapsto (X, X),\ \mathfrak{g}^* \to \mathfrak{d} : f \mapsto (r_+ f, r_- f)$$

Equip with the inner product

$$\langle (X_1, X_2), (Y_1, Y_2) \rangle = \langle X_1, Y_1 \rangle - \langle X_2, Y_2 \rangle$$

where $\langle\ ,\ \rangle$ is the natural inner product on $\mathfrak{g}$. Then $\mathfrak{d}$ is the linear sum of these embedded subalgebras, $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$ and the canonical projections onto $\mathfrak{g}$, $\mathfrak{g}^*$ in this decomposition are adjoints of each other. Note that although of course $\mathfrak{d}^* \simeq \mathfrak{g} \oplus \mathfrak{g}^*$ as a Lie algebra, in standard coordinates on $\mathfrak{g} \oplus \mathfrak{g}$ the dual bracket looks twisted. We may briefly, though somewhat informally, say that the double of a factorizable Lie bialgebra coincides with its twisted square (cf. Theorem 2.9 below).

Conversely, let $\mathfrak{d}$ be a Lie algebra equipped with a nondegenerate invariant inner product and $\mathfrak{a}$, $\mathfrak{b}$ its Lie subalgebras. Assume that $\mathfrak{a}$, $\mathfrak{b}$ are isotropic and the restriction of the inner product to $\mathfrak{a} \times \mathfrak{b}$ is nondegenerate, so that $\mathfrak{a}$, $\mathfrak{b}$ are the duals of each other. Then $(\mathfrak{a}, \mathfrak{b})$ is a Lie bialgebra and $\mathfrak{d}$ is its double. The set $(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ is referred to as the Manin triple.

Example. Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra, $\mathfrak{b}_+ \subset \mathfrak{g}$ a Borel subalgebra, containing $\mathfrak{h}$, $\mathfrak{b}_- \subset \mathfrak{g}$ the opposite Borel subalgebra, $\mathfrak{m}_\pm = [\mathfrak{b}_\pm, \mathfrak{b}_\pm]$. Let $\pi : \mathfrak{b}_\pm \to \mathfrak{b}_\pm / \mathfrak{n}_\pm \simeq \mathfrak{h}$ be the natural projection. Let $(\ ,\ )$ be the Killing form on $\mathfrak{g}$. It induces a natural inner product on $\mathfrak{h}$. Put $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$. We equip $\mathfrak{d}$ with the inner product

$$\langle (X, H), (Y, H') \rangle = (X, Y) - (H, H'). \tag{1.28}$$

Let θ be an orthogonal operator in $\mathfrak{h}$.

We define embeddings $r^\theta_\pm : \mathfrak{b}_\pm \to \mathfrak{d}$ by

$$r^\theta_+ : X_+ \mapsto (X_+, \pi(X_+)), \tag{1.29}$$

$$r_-^\theta : X_- \mapsto (X_-, \theta(\pi(X_-))). \tag{1.29}$$

PROPOSITION 1.6. *Assume that $(id - \theta)$ is invertible. Then the operators $r_\pm^\theta$ define the structure of a Lie bialgebra on $(\mathfrak{b}_+, \mathfrak{b}_-)$ and $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ is its double.* ∎

COROLLARY *The projection $\mathfrak{g} \oplus \mathfrak{f} \to \mathfrak{g}$ induces the structure of factorizable Lie bialgebra on $(\mathfrak{g}, \mathfrak{g}^*)$. Its dual is isomorphic to*

$$\mathfrak{g}^* = \{(X_+, X_-) \in \mathfrak{b}_+ \oplus \mathfrak{b}_-;\ \theta(\pi(X_+)) = \pi(X_-)\}$$ ∎

The associated factorization problem in $\mathfrak{g}$ is

$$X = X_+ - X_- \text{ with } X_\pm \in \mathfrak{b}_\pm, \theta(\pi(X_+)) = \pi(X_-). \tag{1.30}$$

The most common choice of θ is, of course, $\theta = -id$. However, for rank $\mathfrak{g} > 1$ there exists a continuous family of r-matrices on $\mathfrak{g}$. (There is also an additional freedom associated with parabolic subalgebras in $\mathfrak{g}$, see [10, 11]).

2. Let us now turn to the study of the quantum case. Recall that "Quantum groups" are Hopf algebras which may be regarded as deformations of universal enveloping algebras. We start with the definition of factorizable Hopf algebras which is suggested by Proposition 1.3. Our next step will be to prove that such algebras actually exist, which is again achieved by squaring an arbitrary Hopf algebra.

Let A be a Hopf algebra with product $m : A \otimes A \to A$, coproduct $\Delta : A \to A \otimes A$, unit e, counit ϵ and antipode S. We shall denote by Δ' the opposite coproduct obtained from Δ by permutation

$$\Delta'(x) = P(\Delta(x)), \quad P(a \otimes b) = b \otimes a. \tag{2.1}$$

Let A^* be the dual Hopf algebra. Recall that by definition the structure of a Hopf algebra on A^* is defined by the formulae

$$\begin{aligned} &\langle \Delta^* f, x \otimes y\rangle = \langle f, xy\rangle, \\ &\langle fg, x\rangle = \langle f \otimes g, \Delta x\rangle, \\ &\langle Sf, x\rangle = \langle f, Sx\rangle. \end{aligned} \tag{2.2}$$

We denote by A^0 the opposite Hopf algebra of A^* in which the coproduct is given by

$$\langle \Delta^0 f,\ x \otimes y\rangle = \langle f, yx\rangle \tag{2.3}$$

The antipode in A^0 is defined by

$$\langle S'f, x\rangle = \langle f, S^{-1}(x)\rangle.$$

The algebra A is said to be quasitriangular [3] if there is an invertible element $R \in A \otimes A$ satisfying

$$(2.4)\qquad \begin{aligned}(\Delta \otimes id)\, R &= R_{13}\, R_{23}, \\ (id \otimes \Delta)\, R &= R_{13}\, R_{12}\end{aligned}$$

and such that

$$(2.5)\qquad \Delta'(a) = R\, \Delta(a)\, R^{-1}$$

for all $a \in A$.

PROPOSITION 2.1. *Assume that R satisfies (2.4) and moreover $(\epsilon \otimes id) R = (id \otimes \epsilon) R = e$. Then R is invertible and*

$$(2.6)\qquad R^{-1} = (S \otimes id)\, R = (id \otimes S^{-1})\, R$$

Formula (2.6) is connected with the so-called crossing symmetry of quantum R-matrices and in a slightly disguised form was extensively used in the quantum inverse scattering method. We have

$$\begin{aligned}R(S \otimes id)\, R &= (m_{12} \otimes id)\,(id \otimes S \otimes id)\, R_{13}\, R_{23} = \\ &= (m_{12} \otimes id)\,(id \otimes S \otimes id)\,(\Delta \otimes id)\, R = (\epsilon \otimes id)\, R = e.\end{aligned}$$

The second identity is proved in a similar way, using the second formula in (2.4).

PROPOSITION 2.2. *Let (A, R) be a quasitriangular Hopf algebra. Then R satisfies the Yang-Baxter identity*

$$(2.7)\qquad R_{13}\, R_{23}\, R_{12} = R_{12}\, R_{23}\, R_{13}$$

Proof. Formulae (2.4) imply that

$$(2.8)\qquad (\Delta' \otimes id)\, R = R_{23}\, R_{13}, \quad (id \otimes \Delta')\, R = R_{12} R_{13}.$$

Since $(\Delta' \otimes id) = R_{12}\, (\Delta \otimes id) R_{12}^{-1}$, $(id \otimes \Delta') = R_{23}\, (id \otimes \Delta)\, R_{23}^{-1}$ the consistency condition (2.7) must hold. ∎

Any element $B \in A \otimes A$ defines a linear map

$$(2.9)\qquad B : A^* \to A : f \mapsto \langle f \otimes id, B\rangle$$

PROPOSITION 2.3. *An element $R \in A \otimes A$ satisfying (2.4), or, equivalently, (2.8) defines a Hopf algebra homomorphism $R : A^0 \to A$.*

It is sometimes useful to express the intertwining relations (2.5) in an alternative way making the connection with the quantum inverse scattering method more transparent. Recall that hystorically the definition of quasitriangular Hopf algebra was an algebraic refinement of the R-matrix commutation relations between the matrix coefficients of the quantum monodromy matrix. To clarify the connection with QISM let us introduce the canonical element $T \in A \otimes A^*$.

$$T = \sum_i e_i \otimes e^i$$

where $\{ e_i \}$ is a basis in A and $\{e^i\}$ is the dual basis in A^*. Clearly the definition of T does not depend on the choice of the basis.

PROPOSITION 2.3'. *The canonical element T satisfies the following relations*

$$(\Delta \otimes id)\, T = \sum_{j,k} e_j \otimes e_k \otimes e^j e^k = T_1 T_2, \tag{2.10}$$

$$(\Delta' \otimes id)\, T = \sum_{j,k} e_j \otimes e_k \otimes e^k e^j = T_2 T_1,$$

$$R_{12} T_1 T_2 = T_2 T_1 R_{12}, \quad (S \otimes id)\, T \cdot T = 1. \tag{2.11}$$

It is in this form that the intertwining relation (2.5) is commonly used in Quantum inverse scattering method.

Assume that $R \in A \otimes A$ satisfies (2.4) and (2.7). Put $R_- = P(R^{-1}) = P(S \otimes id)\, R$ where P is the permutation map.

PROPOSITION 2.4. R_- satisfies relations (2.4) and (2.7). Moreover, $R_-^{-1} = P(R)$. ■

As in Section 1 we shall sometimes write R_+ instead of R.

Consider now the following sequence of mappings

$$A^0 \xrightarrow{\Delta^0} A^0 \otimes A^0 \xrightarrow{R_+ \otimes R_-} A \otimes A \xrightarrow{m\,(id \otimes S^{-1})} A. \tag{2.12}$$

Clearly, the composition $I: A^0 \to A$ is given by

$$f \to \langle f \otimes id, I\rangle, \quad I = R\, P(R) = R_+ R_-^{-1}. \tag{2.13}$$

We may now finally introduce the definition of factorizable Hopf algebras.

DEFINITION 2.1. *A Hopf algebra A is called factorizable if it is quasitriangular and, moreover, the linear map (2.13) associated with the corresponding R-matrix is invertable.*

Observe that, for a factorizable A, the mapping $A^0 \xrightarrow{\Delta^0} A^0 \otimes A^0 \xrightarrow{R_+ \otimes R_-} A \otimes A$ defines an embedding of A^0 into $A \otimes A$. Below we shall describe a twisted Hopf algebra structure on $A \otimes A$ which makes this embedding a Hopf algebra homomorphism. ∎

Remark. Sometimes R-matrices satisfying the unitarity condition

(2.14) $$R\,P(R) = e$$

are considered (the corresponding Hopf algebras are called triangular). By contrast with Definition 2.1 this means that the operator (2.13) is a rank 1 projection operator.

PROPOSITION 2.5. *Let (A, R) be a factorizable Hopf algebra. Then any element $x \in A$ admits a unique representation*

(2.15) $$x = \sum_i x_+^i\, S^{-1}(x_-^i)$$

with $\Sigma\, x_+^i \otimes x_-^i$ lying in the range of the mapping $(R_+ \otimes R_-)\,\Delta^0 : A^0 \to A \otimes A$. ∎

The following assertion is a close analogue of Proposition 1.4.

PROPOSITION 2.6. *Put*

(2.16) $$x * y = I(I^{-1}(x) \cdot I^{-1}(y)), \quad x, y \in A.$$

Then

(2.17) $$x * y = \sum_i x_+^i\, y\, S^{-1}(x_-^i).$$ ∎

Thus the formal properties of factorizable Hopf algebras are the same as in the quasiclassical case considered in Section 1. The main difference is of course that generically A contains only very few group-like elements, and hence the sum in (2.15) cannot be eliminated (cf. (1.23), (1.24)).

Let us now turn to the construction of the double of a Hopf algebra. We start with the notion of twisted product of Hopf algebras.

Let A, B be arbitrary Hopf algebras. Assume that there is an element $R \in B \otimes A$

such that

$$(\Delta_B \otimes id)R = R_{23} R_{13},$$

(2.18) $$(id \otimes \Delta_A)R = R_{12} R_{13},$$

$$(id \otimes S_A)R = R^{-1}, \quad (S_B \otimes S_A)R = R$$

Here Δ_A, Δ_B, S_A, S_B are the coproducts and antipodes in A, B, respectively. We define twisted coproduct on the algebra $A \otimes B$ by

(2.19) $$\Delta(x \otimes y) = R_{23}\, \Delta^A_{13}(x)\, \Delta^B_{24}(y)\, R^{-1}_{23}$$

THEOREM 2.7. *Formula (2.20) defines the structure of a Hopf algebra on $A \otimes B$ and*

(2.20) $$S(x \otimes y) = P(R)^{-1}\, S_A(x) \otimes S_B(y)\, P(R)$$

is its antipode. (Here P is the permutation operator mapping $B \otimes A$ to $A \otimes B$). The Hopf algebra described in Theorem 2.7 will be called the twisted product of A and B and denoted $A \otimes_R B$.

Proof. The proof of the coassociativity of (2.19) is by direct computation based on the following property of the R-matrix implied by (2.4)

$$(\Delta \otimes id)R \cdot R = R_{13} R_{23} R_{12},$$

$$R\,(id \otimes \Delta)R = R_{13} R_{23} R_{12}.$$

Let us check the formula (2.20) for the antipode. We have first of all

$$\Delta(S(x \otimes y)) = \Delta(P(R)^{-1}\, S^A(x) \otimes S^B(y)\, P(R)) =$$

$$\Delta(P(R))\, R_{23} S^A_1 S^B_2 S^A_3 S^B_4 (\Delta^A_{31}(x)\, \Delta^B_{42}(y))\, R^{-1}_{23}\, \Delta(P(R)) =$$

$$R_{23}\, \Delta^A_{13}\, \Delta^B_{24}(P(R)^{-1}) S^A_1 S^B_2 S^A_3 S^B_4\, (\Delta^A_{31}(x)\, \Delta^B_{42}(y))$$

$$\cdot\, \Delta^A_{13}\ \Delta^B_{24}\, (P(R))\, R^{-1}_{23}\ .$$

On the other hand,

$$\Delta'(x \otimes y) = R_{41}\, \Delta^A_{31}(x)\, \Delta^B_{42}(y)\, R^{-1}_{41},$$

$$(S \otimes S)\Delta'(x \otimes y) = P(R)^{-1}_{12}\ P(R)^{-1}_{34}\ S^A_1 S^B_2 S^A_3 S^B_1\ (\Delta'(x \otimes y)) \cdot$$

$$\cdot\, P(R)_{12}\, P(R)_{34} =$$

$$P(R)^{-1}_{12} P(R)^{-1}_{34}\ S^B_4 S^A_1 (R^{-1}_{41}) S^A_1 S^B_2 S^A_3 S^B_4\ (\Delta^A_{31}(x)\, \Delta^B_{42}(y)) \cdot$$

$$S^B_4 S^A_1\, (R_{41}) P(R)_{12}\ P(R)_{34}.$$

By comparing these expressions we find that

$$\Delta(S(x \otimes y)) = (S \otimes S)\,(\Delta'(x \otimes y))$$

provided that

$$R_{23}\Delta^A_{13}\Delta^B_{24}\; P(R)^{-1} = P(R)^{-1}_{12}\, P(R)^{-1}_{34}\, S^B_4 S^A_1\, (R^{-1}_{41}).$$

This relation holds if

$$(S^B \otimes S^A)\, R = R.$$

Let us now check that

$$m(S \otimes id)\,\Delta = \epsilon.$$

Put $R = \alpha_i \otimes \beta_i$, $R^{-1} = \bar{\beta}_i \otimes \bar{\alpha}_i$ (we shall perform summation over repeated indices). We have

$$m(S \otimes id)\,\Delta(x \otimes y) = m\,(S \otimes id)\,(R_{23}\;\Delta^A_{13}(x)\Delta^B_{24}(y)\,R^{-1}_{23}) =$$

$$m(R^{-1}_{21}\; S^A_1\; S^B_2\; (R_{23}\;\Delta^A_{13}(x)\,\Delta^B_{24}(y)\,R^{-1}_{23})\,R_{21}).$$

Let

$$\Delta^A(x) = x^i \otimes x_i,\; \Delta^B(y) = y^i \otimes y_i.$$

Then the l.h.s. of the above expression is equal to

$$m(\bar{\alpha}_n\; S_A\; (x^j)\,\alpha_m \otimes \bar{\beta}_n\; S^B(\beta_\ell)\, S^B(y_k)\, S^B(\beta_i)\,\beta_m$$

$$\alpha_i\, x_j\, \bar{\alpha}_\ell \otimes y_k) =$$

$$\bar{\alpha}_n\; S^A(x^j)\,\alpha_m\,\alpha_i\, x_j\, \bar{\alpha}_\ell \otimes \bar{\beta}_n S^B(\beta_\ell)\, S^B(y^k)\, S^B(\beta_i)\,\beta_m\, y_k.$$

Now, since $(id \otimes S^A)\,R = R^{-1}$ we have

$$\alpha_m\alpha_i \otimes S^B\;(\beta_i)\,\beta_m = 1 \otimes 1$$

and also

$$\bar{\alpha}_n\;\bar{\alpha}_\ell \otimes \bar{\beta}_n\; S^B(\bar{\beta}_\ell) = 1 \otimes 1$$

since $(S^B \otimes id)R^{-1} = R$ which concludes the argument. ■

Now, let A be an arbitrary Hopf algebra, A' the opposite of A, i.e. it is endowed with the product $m'(x \otimes y) = yx$ and with the same coproduct. Let A^* be the dual of A. Let R be the canonical element in $A^* \otimes A'$ defined in the following (fairly standard) way. Fix a basis $\{e_i\}$ in A' and a dual basis $\{e^i\}$ in A^* and put

(2.21) $$R = \sum_i e^i \otimes e_i.$$

LEMMA. *R satisfies relations (2.18).* ■

Now, put $T(A) = A' \otimes_R A^*$, and let $\mathscr{D}(A) = T(A)^*$ be its dual. Clearly, $\mathscr{D}(A) \simeq A^0 \otimes A$ as a linear space.

THEOREM 2.8. *(i) Canonical embeddings $A^0, A \to \mathscr{D}(A)$ are Hopf algebra homomorphisms. (ii) $\mathscr{D}(A) = A^0 \otimes A$ as a coalgebra. (iii) Let $R_D \in A^0 \otimes A \subset \mathscr{D}(A) \otimes \mathscr{D}(A)$ be the canonical element. Then*

(2.22) $$\Delta'_{\mathscr{D}}(x \otimes y) = R_{\mathscr{D}}\, \Delta_{\mathscr{D}}\, (x \otimes y)\, R_{\mathscr{D}}^{-1}$$

In other words, $\mathscr{D}(A)$ is quasitriangular. ■

Multiplication in $\mathscr{D}(A)$ is dual to (2.19) with R given by (2.21) and is given by rather cumbersome formulae. Let us choose a basic $\{e_i\}$ in A and a dual basic $\{e^i\}$ in A^0 and let

(2.23) $$\begin{gathered} e_i e_j = m(e_i \otimes e_j) = m_{ij}^k\, e_k, \quad \Delta e_i = \mu_i^{jk}\, e_j \otimes e_k, \\ S^{-1}(e_i) = (S^{-1})_i^j\, e_j \end{gathered}$$

(Here and below we perform summations over repeated indices). By directly applying the definitions one can show that the following commutation relations for the multiplication in $\mathscr{D}(A)$ are valid

(2.24) $$\begin{gathered} e_s e^t = \mu_s^{jn} (S^{-1})_n^p\, m_{pa}^t\, m_{qk}^a\, \mu_j^{k\ell}\, e^q e_\ell, \\ e^i e_j = \mu_j^{sn}\, (S^{-1})_n^v\, m_{vr}^i\, \mu_s^{pq}\, m_{qt}^r\, e_p\, e^t \end{gathered}$$

Algebra $\mathscr{D}(A)$ is called the double of A. It was constructed by Drinfeld in [3]; the coordinate-free description we give here is new.

If A is a factorizable Hopf algebra, we may give an alternative description of its double which parallels Proposition 1.5.

THEOREM 2.9. *Let (A, R) be a factorizable Hopf algebra. Then its double $\mathscr{D}(A)$ coincides with its twisted square $A \otimes_R A$. In other words, $\mathscr{D}(A) = A \otimes A$ is an algebra and the coproduct in $\mathscr{D}(A)$ is given by*

(2.25) $$\Delta\ (x \otimes y) = R_{23}^{-1}\, \Delta_{13}(x)\, \Delta_{24}(y)\, R_{23}.$$

Sketch of a proof. Embed $A, A^0 \to A \otimes A$ via the mappings Δ, $(R_+ \otimes R_-)\Delta^0$

which were used to define the factorization in A. One checks immediately that to make these embeddings Hopf algebra homomorphisms, one has to twist the coproduct in $A \otimes A$ as in (2.25). Theorem 2.7 now assures that (2.25) is indeed a coassociative coproduct in $A \otimes A$ ∎

Let $R_{\mathscr{D}}\cdot$ be the image of the canonical element $e_i \otimes e^i \in A \otimes A^0$ under the ambedding $A \otimes A^0 \to A \otimes A \otimes A \otimes A$. We have

(2.26) $$R_{\mathscr{D}} = \sum_i \Delta_{13}\, e_i \otimes (R_+ \otimes R_-)\, \Delta^0_{24}\, e^i = R^{(-)}_{14} R^{(-)}_{24} R_{13} R_{23}$$

with $R^{(-)} = PR^{-1}$. It is a direct calculation to show that $\mathscr{D} = A \otimes_R A$ is quasi-triangular with the R-matrix $R_{\mathscr{D}}$ given by (2.26).

THEOREM 2.10. *The double of an arbitrary Hopf algebra A is factorizable.*

Proof. Choose a basis $\{e_i\}$ in A and a dual basis $\{e^i\}$ in A^0. Elements $f_i^j = e^j e_i$ form a linear basis in $\mathscr{D}(A) \simeq A^0 \otimes A$. Let $\{\varphi_k^\ell\}$ be the dual basis in $T(A) = \mathscr{D}(A)^*$:

(2.27) $$\langle \varphi_k^\ell, f_i^j \rangle = \delta_i^{\ \ell} \delta_k^{\ j}$$

The R-matrix $R_{\mathscr{D}}$ is given by

(2.28) $$R_{\mathscr{D}} = e_i \otimes e^i \in \mathscr{D} \otimes \mathscr{D}$$

and hence

(2.29) $$I = R_{\mathscr{D}} \cdot P(R_{\mathscr{D}}) = e_i e^j \otimes e^i e_j$$

The linear operator defined by (2.29) is given by

$$I : \varphi_k^{\ \ell} \to e_k e^\ell$$

i.e. its acts simply as a reordering. In view of the commutation relations (2.24) it is clear that I is invertible. ∎

Example. Let $\mathscr{U}_q(\mathfrak{g})$ be the quantized universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$ [3] and $\mathscr{U}_q(\mathfrak{b}_+)$ ($\mathscr{U}_q(\mathfrak{b}_-)$) its subalgebra corresponding to the Borel subalgebra $\mathfrak{b}_+$ ($\mathfrak{b}_-$). (It is known [3] that $\mathscr{U}_q(\mathfrak{b}_+)^0 \simeq \mathscr{U}_q(\mathfrak{b}_-)$ and $\mathscr{D}(\mathscr{U}_q(\mathfrak{b}_+)) \simeq \mathscr{U}_q(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{h})$ as an algebra, where $\mathfrak{h} \subset \mathfrak{b}_+$ is the Cartan subalgebra.

Note that the canonical coproduct on $\mathscr{D}(\mathscr{U}_q(\mathfrak{b}_+))$ is again obtained from the coproduct on the tensor product of Hopf algebras $\mathscr{U}_q(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{h})$ by twisting with the help of the R-matrix $R_0 \in \mathscr{U}(\mathfrak{h}) \otimes \mathscr{U}(\mathfrak{h})$ which is defined as follows.

Let $\{H_i\}$ be the orthogonal basis in $\mathfrak{h}$. Then, by definition,

$$R_0 = \left(exp \sum_i H_i \quad H_i\right).$$

Since $\mathcal{D}$ ($\mathcal{U}_q$($\mathfrak{b}_+$) is factorizable we immediately obtain the following result.

PROPOSITION 2.11. *The quantized universal enveloping algebras of simple Lie algebras are factorizable.* ■

3. DRESSING TRANSFORMATIONS

To motivate the definitions, we start again with the classical case. Let ($\mathfrak{g}$, $\mathfrak{g}^*$) be a Lie bialgebra, G, G^* the corresponding Poisson Lie group (we shall always have in mind local Poisson groups). Let ($\mathfrak{d}$, $\mathfrak{d}^*$) be the double of ($\mathfrak{g}$, $\mathfrak{g}^*$). The corresponding Lie group $\mathcal{D}$ is called the double of G; it is again a Poisson Lie group and its tangent Lie bialgebra coincides with ($\mathfrak{d}$, $\mathfrak{d}^*$). $\mathcal{D}$ contains both G, G^* as subgroups, and as a Poisson manifold (though of course not as a group) $\mathcal{D} \simeq G \times G^*$. In particular, the quotient space $\mathcal{D}/G^*$ (consisting of left coset classes) may be identified with G.

PROPOSITION 3.1. *(i) The ring of right G^*-invariant functions on $\mathcal{D}$ is a Lie subalgebra with respect to the Poisson bracket on $\mathcal{D}$. (ii) Restriction of right – G^* – invariant functions to $G \subset \mathcal{D}$ induces an isomorphism of Lie algebras*

$$C^\infty(\mathcal{D})^{G^*} \to C^\infty(G).$$ ■

This is an example of the Poisson reduction technique discussed in [9]. The group $\mathcal{D}$ acts on $C^\infty(\mathcal{D})$ by left translations. Clearly, it leaves $C^\infty(\mathcal{D})^{G^*}$ invariant. Restricting this action to the subgroup $G^* \subset \mathcal{D}$ and combining it with the above isomorphism we get a linear action

$$G^* \times C^\infty(G) \to C^\infty(G) \tag{3.1}$$

This action induces an action of G^* on the group G itself; it is this latter action that is usually referred to as dressing transformations. We have

$$h : g \to (hg)_+ \tag{3.2}$$

where the product is in $\mathcal{D}$ and $(hg)_+$ denotes the solution to the factorization problem in $\mathcal{D}$

$$x = x_+ x_-^{-1}, \quad x_+ \in G, \ x_- \in G^* \tag{3.3}$$

with $x = hg$.

THEOREM 3.2. *The mapping $G^* \times G \to G$ defined above is a morphism of Poisson manifolds.* ■

In other words, formula (3.1) defines a Poisson group action.

Since the roles of G and G^* are completely symmetric we may also define the dual action

$$G \times G^* \to G^*$$

One can easily see that when $(\mathfrak{g}, \mathfrak{g}^*)$ is a trivial Lie bialgebra (i.e. $\mathfrak{g}^*$ is abelian and hence $G^* \simeq \mathfrak{g}^*$) the action $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ coincides with the coadjoint representation of G. In many ways dressing action may be regarded as an appropriate generalization of the coadjoint representation.

Let us now turn to the quantum case. Let A be a Hopf algebra and $\mathscr{D}(A)$ its double described in Theorem 2.8. Let T be the dual of $\mathscr{D}(A)$, and T^{A^*} its subalgebra consisting of right A^*-invariant functionals,

$$T^{A^*} = \{ f \in T; \langle f, xa \rangle = \epsilon(a) \langle f, x \rangle \} \tag{3.4}$$

PROPOSITION 3.3. *Restriction of right $-A^*-$ invariant functionals to $A \subset \mathscr{D}(A)$ induces an isomorphism of Hopf algebras*

$$T^{A^*} \simeq A^*$$ ■

Clearly, T^{A^*} is invariant with respect to the action of A^* on the left

$$\langle a \cdot f, x \rangle = \langle f, ax \rangle \tag{3.5}$$

where the multiplication on the r.h.s. is in $\mathscr{D}(A)$. In the dual way, we get an action

$$A^* \otimes A \to A : a \otimes x \to (id \otimes \epsilon) \sum y^i_+ \otimes y^i_- \tag{3.6}$$

where

$$ax = \sum_i y^i_+ \, S^{-1} \, y^i_-$$

is the solution of the factorization problem in $\mathscr{D}(A)$.

(To understand properly the analogy between classical and quantum cases we must keep in mind that neither A nor A^* is an analog of the group G itself; rather the elements of A^* are analogs of functions on G and those of A are analogs of differential operators on G. In other words, a Quantum group does not have any "points").

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